1. The curve C has equation $y = f(x)$ where

$$f(x) = ax^3 + 15x^2 - 39x + b$$

and a and b are constants.

Given

- the point (2, 10) lies on C
- the gradient of the curve at (2, 10) is –3

(a) (i) show that the value of a is –2

(ii) find the value of b.

(4)

(b) Hence show that C has no stationary points.

(3)

(c) Write f(x) in the form $(x - 4)Q(x)$ where $Q(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = f(0.2x)$$

and the coordinate axes.

(2)

(a)

(i) $\frac{dy}{dx} = 3ax^2 + 30x - 39$

$\frac{dy}{dx} = -3$ when $x = 2$ $\Rightarrow -3 = 3a(2)^2 + 30(2) - 39$ (1)

$12a = -24$

$a = -2$ (1)

(ii) As $f(2) = 10$

$\Rightarrow (-2)(2)^3 + 15(2)^2 - 39(2) + b = 10$ (1)

$-16 + 60 - 78 + b = 10$

$b = 44$ (1)

(b) $f'(x) = -6x^2 + 30x - 39$ (1)

$b^2 - 4ac \Rightarrow 30^2 - 4(-6)(-39) = -36 < 0$

(1)

Since $b^2 - 4ac < 0$, $f'(x) \neq 0$ so no stationary point exists. (1)

c) $f(x) = -2x^3 + 15x^2 - 39x + 44$

$f(x) = (x-4)\,Q(x)$

$$
\begin{array}{r|l}
 & -2x^2 + 7x - 11 \quad 1 \\
\hline
x-4 & -2x^3 + 15x^2 - 39x + 44 \\
 & -\;(-2x^3 + 8x^2) \\
\hline
 & 7x^2 - 39x \\
 & -\;(7x^2 - 28x) \\
\hline
 & -11x + 44 \\
 & -\;(-11x + 44) \\
\hline
\end{array}
$$

$f(x) = (x-4)(-2x^2 + 7x - 11)$ 1

d) when $x = 0$, $f(0) = f(0.2 \times 0) = 44$

$(0, 44)$

when $y = 0$

$(0.2x - 4)(-2 \times (0.2x)^2 + 7(0.2x) - 11) = 0$

$\Rightarrow 0.2x - 4 = 0$

$x = 20$

$(20, 0)$

1st term 2nd term

$f(0.2x) = (0.2x - 4)(-0.08x^2 + 1.4x - 11) = 0$

$x = 20$ is the only solution to $f(0.2x) = 0$ since 2nd term is < 0 when we put into $b^2 - 4ac \Rightarrow 1.4^2 - 4(-0.08)(-11) = -1.56$

$f(0.2x)$ intersects at y-axis

$f(0.2x)$ intersects x-axis

Point of intersection : $(0, 44)$ 1 and $(20, 0)$ 1

2. (a) Factorise completely $9x - x^3$

(2)

The curve C has equation

$$y = 9x - x^3$$

(b) Sketch C showing the coordinates of the points at which the curve cuts the x-axis.

(2)

The line l has equation $y = k$ where k is a constant.

Given that C and l intersect at 3 distinct points,

(c) find the range of values for k, writing your answer in set notation.

Solutions relying on calculator technology are not acceptable.

(3)

a) $9x - x^3 \equiv x(9 - x^2)$ 1

$\equiv x(3+x)(3-x)$ 1

b) $y = 9x - x^3$

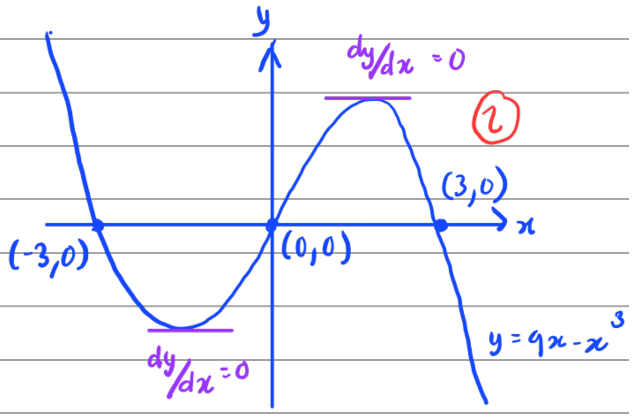
when $x = 0$, $y = 9(0) - (0)^3 = 0$

when $y = 0$, $0 = 9x - x^3$

$0 = x(3+x)(3-x)$

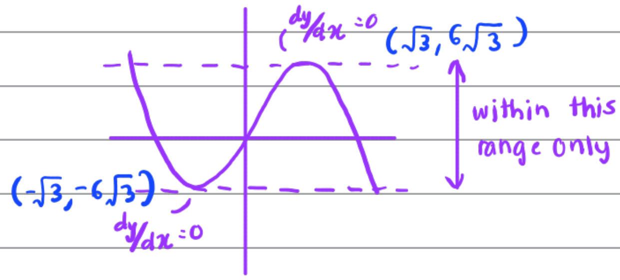
$x = -3, 0, 3$

Curve C:



c) For line l to intersect at 3 points of curve C, the intersection points can only be within 2 turning points of the curve

Turning points : $\frac{dy}{dx} = 0$



$\frac{dy}{dx} = 9 - 3x^2$

$\therefore 9 - 3x^2 = 0$ (1)

$x^2 = 3$

$x = \pm\sqrt{3}$

when $x = \sqrt{3}$, $y = 9\sqrt{3} - (\sqrt{3})^3 = 6\sqrt{3}$ (1)

$x = -\sqrt{3}$, $y = 9(-\sqrt{3}) - (-\sqrt{3})^3 = -6\sqrt{3}$

$\therefore \{k \in \mathbb{R} : -6\sqrt{3} < k < 6\sqrt{3}\}$ (1)

3. A curve has equation $y = f(x),\ x \geqslant 0$

Given that

- $f'(x) = 4x + a\sqrt{x} + b$, where a and b are constants
- the curve has a stationary point at $(4, 3)$
- the curve meets the y-axis at -5

find $f(x)$, giving your answer in simplest form.

(6)

when curve is at stationary point, $f'(x) = 0$, $x = 4$ and $y = 3$.

$f'(x) = 4x + a\sqrt{x} + b$

$0 = 4(4) + a\sqrt{4} + b$

$0 = 16 + 2a + b$ (1) , $b = -2a - 16$ — (1)

To get $f(x)$, we will integrate $f'(x)$.

$f'(x) = 4x + a\sqrt{x} + b$ (1)

$f(x) = 2x^2 + \frac{2}{3}ax^{\frac{3}{2}} + bx + c$ (1)

y-intercept is -5, so the value of $c = -5$ (1)

$f(x) = 2x^2 + \frac{2}{3}ax^{\frac{3}{2}} + bx - 5$

from the stationary point, we know that $f(4) = 3$

$3 = 2(4)^2 + \frac{2}{3}a(4)^{\frac{3}{2}} + 4b - 5$

$3 = 32 + \frac{16}{3}a + 4b - 5$

$4b = -24 - \frac{16}{3}a$

$b = -6 - \frac{16}{12}a$ — (2)

substitute ① into ②

$$-2a - 16 = -6 - \frac{16}{12}a$$

$$-2a + \frac{4}{3}a = 10$$

$$-\frac{2}{3}a = 10$$

$$a = -15 \quad , \quad b = 14 \quad (1)$$

$$f(x) = 2x^2 + \frac{2}{3}(-15)x^{3/2} + 14x - 5$$

$$= 2x^2 - 10x^{\frac{3}{2}} + 14x - 5 \quad (1)$$

4. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a) (i) $y = 5x^4 - 24x^3 + 42x^2 - 32x + 11$

$\dfrac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$ (1)

(1)

(ii) $\dfrac{d^2y}{dx^2} = 60x^2 - 144x + 84$ (1)

b) (i) If C has a stationary point at $x=1$, then $\left.\dfrac{dy}{dx}\right|_{x=1} = 0$

$\left.\dfrac{dy}{dx}\right|_{x=1} = 20(1)^3 - 72(1)^2 + 84(1) - 32$

$= 20 - 72 + 84 - 32 = 0$ ✓ (1)

so there is a stationary point at $x=1$ (1)

(ii) $\left.\dfrac{d^2y}{dx^2}\right|_{x=0.8} = 7.2 > 0$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=1.2} = -2.4 < 0$ (1)

Since there is a change in sign, $x=1$ is a point of inflection. (1)

5.

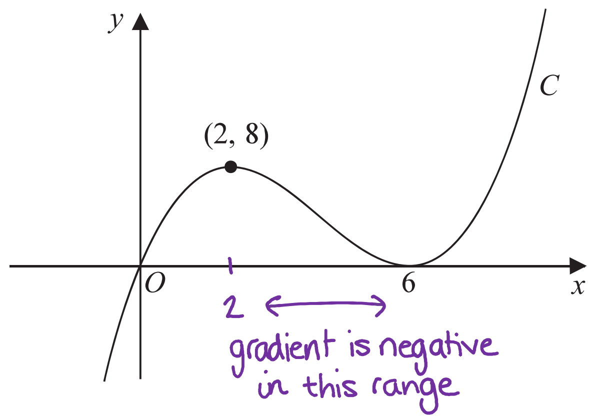


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is a cubic expression in x.

The curve

- passes through the origin
- has a maximum turning point at $(2, 8)$
- has a minimum turning point at $(6, 0)$

(a) Write down the set of values of x for which

$$\mathrm{f}'(x) < 0$$

(1)

The line with equation $y = k$, where k is a constant, intersects C at only one point.

(b) Find the set of values of k, giving your answer in set notation.

(2)

(c) Find the equation of C. You may leave your answer in factorised form.

(3)

(a) $2 < x < 6$ 1

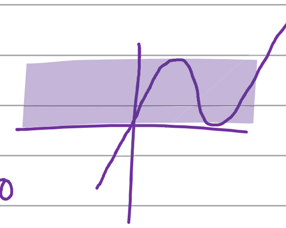
$f'(x) < 0$ means the gradient is negative.
Negative gradient = line going down. ↘

(b) $k > 8$ or $k < 0$ 1

$\{k : k > 8\} \cup \{k : k < 0\}$ 1

y=k is a horizontal line through the y-axis.

has to be outside of shaded region to intersect only once. →



CHOOSE ONE OF THESE METHODS.

(c) Method 1 : Recognise curve has form $y = ax(x-6)^2$ ① states form of c

$(2,8) \rightarrow \quad 8 = 2a(2-6)^2$ ①

$8 = 32a$

$a = \frac{1}{4}$

$\therefore y = \frac{1}{4}x(x-6)^2$ ①

Method 2 : Solving Simultaneous Equations

$y = ax^3 + bx^2 + cx$ ← no $+d$ because the curve goes through the origin.

when $x = 2$, $y = 8$:

$8 = a(2^3) + b(2^2) + c(2)$

① $4 = 4a + 2b + c$

when $x = 6$, $y = 0$:

$0 = a(6^3) + b(6^2) + c(6)$

② $0 = 216a + 36b + 6c$ ① for 2 sim. eq.

$f'(x) = 3ax^2 + 2bx + c$

when $x = 6$, $f'(x) = 0$: ← $(6,0)$ is a turning point

$0 = 3a(6^2) + 2b(6) + c$

③ $0 = 108a + 12b + c$

Solve ①, ②, ③ simultaneously : ← use a calculator or solve by hand.

$4 = 4a + 2b + c$

$0 = 216a + 36b + 6c$

$0 = 108a + 12b + c$

$a = \frac{1}{4}, \quad b = -3, \quad c = 9$ ① for solving sim. eq.

$y = \frac{1}{4}x^3 - 3x^2 + 9x$ ①

6.

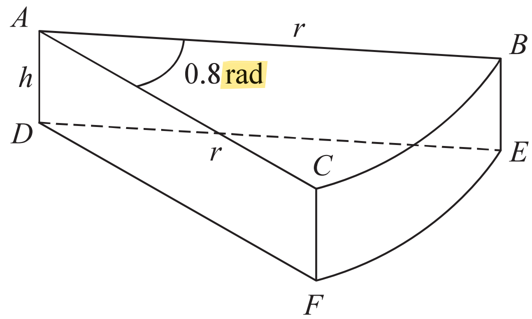


Figure 5

A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD, CF and BE are perpendicular to faces ABC and DEF
- edges AD, CF and BE have length h cm

Given that the volume of the toy is 240 cm^3

(a) show that the surface area of the toy, $S\text{ cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

(2)

(a) $\underbrace{\frac{1}{2} \times 0.8 \times r^2}_{\text{area of sector}} \times \underbrace{h}_{\text{height}} = \underbrace{240}_{\text{volume}}$ 1 $\qquad \frac{1}{2}\theta r^2$ = area of sector (when θ is in radians)

$0.4r^2h = 240$ $\div 0.4$

$r^2h = 600$ $\div r^2$

$h = \frac{600}{r^2}$ 1

total surface area = 2 × $\overset{\text{area of}}{\text{sector face}}$ + 2 × $\overset{\text{area of}}{\text{sector length}}$ + $\overset{\text{area of}}{\text{arc}}$

$$S = 2\left(\tfrac{1}{2}\theta r^2\right) + 2(rh) + (r\theta \times h)$$

$$S = 0.8r^2 + 2rh + 0.8rh$$

$h = \dfrac{600}{r^2}$

$$S = 0.8r^2 + 2r\left(\frac{600}{r^2}\right) + 0.8r\left(\frac{600}{r^2}\right) \quad 1$$

$$S = 0.8r^2 + \frac{1200}{r} + \frac{480}{r}$$

$$S = 0.8r^2 + \frac{1680}{r} \quad 1$$

(b) $S = 0.8r^2 + 1680r^{-1}$ $\quad \frac{1}{x} = x^{-1}$

$$\frac{dS}{dr} = 0.8 \times 2r^{2-1} + (-1) \times 1680r^{-1-1}$$

$$= 1.6r - 1680r^{-2} \quad 2$$

$$0 = 1.6r - \frac{1680}{r^2} \quad 1$$

← $\frac{dS}{dr} = 0$ at stationary point

$$1.6r = \frac{1680}{r^2}$$

$\times r^2$

$$1.6r^3 = 1680$$

$\div 1.6$

$$r^3 = 1050$$

$\sqrt[3]{\ }$

$$r = \sqrt[3]{1050}$$

$$r = 10.16 \quad 1$$

(c) $\frac{dS}{dr} = 1.6r - 1680r^{-2}$

$\frac{d^2S}{dr^2} = 1.6 \times 1r^{1-1} - (-2) \times 1680r^{-2-1}$

$= 1.6 + 3360r^{-3}$

$= 1.6 + \frac{3360}{r^3}$

when $r = 10.16$: ← from part (b), stationary point at $r = 10.16$

$1.6 + \frac{3360}{(10.16)^3} = 4.80$ (1)

$\frac{d^2S}{dr^2} > 0$ when $r = 10.16$ therefore this is a minimum value of S. (1)